

# V-SEMI-SLANT SUBMERSIONS

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ABSTRACT. Let  $F$  be a Riemannian submersion from an almost Hermitian manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$ . We introduce the notion of the v-semi-slant submersion. And then we obtain some properties on it. In particular, we give some examples for it.

## 1. INTRODUCTION

Let  $F$  be a  $C^\infty$ -submersion from a semi-Riemannian manifold  $(M, g_M)$  onto a semi-Riemannian manifold  $(N, g_N)$ . Then according to the conditions on the map  $F : (M, g_M) \mapsto (N, g_N)$ , we have the following submersions:

semi-Riemannian submersion and Lorentzian submersion [7], Riemannian submersion ([8], [14]), slant submersion ([5], [19]), almost Hermitian submersion [21], contact-complex submersion [9], quaternionic submersion [10], almost h-slant submersion and h-slant submersion [15], semi-invariant submersion [20], almost h-semi-invariant submersion and h-semi-invariant submersion [16], semi-slant submersions [18], almost h-semi-slant submersions and h-semi-slant submersions [17], etc. As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([3], [22]), Kaluza-Klein theory ([2], [11]), Supergravity and superstring theories ([12], [13]), etc.

The paper is organized as follows. In section 2 we remind some notions which is needed for this paper. In section 3 we give the definition of the v-semi-slant submersion and obtain some interesting properties on it. In section 4 we construct some examples of the v-semi-slant submersion.

## 2. PRELIMINARIES

Let  $(M, g_M)$  and  $(N, g_N)$  be Riemannian manifolds, where  $M, N$  are  $C^\infty$ -manifolds and  $g_M, g_N$  are Riemannian metrics, and  $F : M \mapsto N$  a  $C^\infty$ -submersion. The map  $F$  is said to be *Riemannian submersion* if the differential  $F_*$  preserves the lengths of horizontal vectors [10].

Let  $(M, g_M, J)$  be an almost Hermitian manifold, where  $J$  is an almost complex structure. A Riemannian submersion  $F : (M, g_M, J) \mapsto (N, g_N)$  is called a *slant submersion* if the angle  $\theta = \theta(X)$  between  $JX$  and the space  $\ker(F_*)_p$  is constant for any nonzero  $X \in T_p M$  and  $p \in M$  [19].

We call  $\theta$  a *slant angle*.

A Riemannian submersion  $F : (M, g_M, J) \mapsto (N, g_N)$  is called a *semi-invariant submersion* if there is a distribution  $\mathcal{D}_1 \subset \ker F_*$  such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad J(\mathcal{D}_1) = \mathcal{D}_1, \quad J(\mathcal{D}_2) \subset (\ker F_*)^\perp,$$

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2000 *Mathematics Subject Classification.* 53C15; 53C43.

*Key words and phrases.* Riemannian submersion and slant angle and totally geodesic.

where  $\mathcal{D}_2$  is the orthogonal complement of  $\mathcal{D}_1$  in  $\ker F_*$  [19].

A Riemannian submersion  $F : (M, g_M, J) \mapsto (N, g_N)$  is called a *semi-slant submersion* if there is a distribution  $\mathcal{D}_1 \subset \ker F_*$  such that

$$\ker F_* = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle  $\theta = \theta(X)$  between  $JX$  and the space  $(\mathcal{D}_2)_p$  is constant for nonzero  $X \in (\mathcal{D}_2)_p$  and  $p \in M$ , where  $\mathcal{D}_2$  is the orthogonal complement of  $\mathcal{D}_1$  in  $\ker F_*$ .

We call the angle  $\theta$  a *semi-slant angle*.

Let  $(M, g_M)$  and  $(N, g_N)$  be Riemannian manifolds and  $F : (M, g_M) \mapsto (N, g_N)$  a smooth map. The second fundamental form of  $F$  is given by

$$(\nabla F_*)(X, Y) := \nabla_X^F F_* Y - F_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM),$$

where  $\nabla^F$  is the pullback connection and we denote conveniently by  $\nabla$  the Levi-Civita connections of the metrics  $g_M$  and  $g_N$  [4]. Recall that  $F$  is said to be *harmonic* if  $\text{trace}(\nabla F_*) = 0$  and  $F$  is called a *totally geodesic* map if  $(\nabla F_*)(X, Y) = 0$  for  $X, Y \in \Gamma(TM)$  [4].

### 3. V-SEMI-SLANT SUBMERSIONS

**Definition 3.1.** Let  $(M, g_M, J)$  be an almost Hermitian manifold and  $(N, g_N)$  a Riemannian manifold. A Riemannian submersion  $F : (M, g_M, J) \mapsto (N, g_N)$  is called a *v-semi-slant submersion* if there is a distribution  $\mathcal{D}_1 \subset (\ker F_*)^\perp$  such that

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle  $\theta = \theta(X)$  between  $JX$  and the space  $(\mathcal{D}_2)_p$  is constant for nonzero  $X \in (\mathcal{D}_2)_p$  and  $p \in M$ , where  $\mathcal{D}_2$  is the orthogonal complement of  $\mathcal{D}_1$  in  $(\ker F_*)^\perp$ .

We call the angle  $\theta$  a *v-semi-slant angle*.

*Remark 3.2.* Let  $F$  be a v-semi-slant submersion from an almost Hermitian manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$ . Then there is a distribution  $\mathcal{D}_1 \subset (\ker F_*)^\perp$  such that

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle  $\theta = \theta(X)$  between  $JX$  and the space  $(\mathcal{D}_2)_p$  is constant for nonzero  $X \in (\mathcal{D}_2)_p$  and  $p \in M$ , where  $\mathcal{D}_2$  is the orthogonal complement of  $\mathcal{D}_1$  in  $(\ker F_*)^\perp$ .

If  $\mathcal{D}_2 = (\ker F_*)^\perp$ , then we call the map  $F$  a *v-slant submersion* and the angle  $\theta$  *v-slant angle* [19]. Otherwise, if  $\theta = \frac{\pi}{2}$ , then we call the map  $F$  a *v-semi-invariant submersion* [20].

Let  $F : (M, g_M, J) \mapsto (N, g_N)$  be a v-semi-slant submersion. Then there is a distribution  $\mathcal{D}_1 \subset (\ker F_*)^\perp$  such that

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle  $\theta = \theta(X)$  between  $JX$  and the space  $(\mathcal{D}_2)_p$  is constant for nonzero  $X \in (\mathcal{D}_2)_p$  and  $p \in M$ , where  $\mathcal{D}_2$  is the orthogonal complement of  $\mathcal{D}_1$  in  $(\ker F_*)^\perp$ .

Then for  $X \in \Gamma((\ker F_*)^\perp)$ , we have

$$X = PX + QX,$$

where  $PX \in \Gamma(\mathcal{D}_1)$  and  $QX \in \Gamma(\mathcal{D}_2)$ .

For  $X \in \Gamma(\ker F_*)$ , we get

$$JX = \phi X + \omega X,$$

where  $\phi X \in \Gamma(\ker F_*)$  and  $\omega X \in \Gamma((\ker F_*)^\perp)$ .

For  $Z \in \Gamma((\ker F_*)^\perp)$ , we obtain

$$JZ = BZ + CZ,$$

where  $BZ \in \Gamma(\ker F_*)$  and  $CZ \in \Gamma((\ker F_*)^\perp)$ .

For  $U \in \Gamma(TM)$ , we have

$$U = \mathcal{V}U + \mathcal{H}U,$$

where  $\mathcal{V}U \in \Gamma(\ker F_*)$  and  $\mathcal{H}U \in \Gamma((\ker F_*)^\perp)$ .

Then

$$\ker F_* = B\mathcal{D}_2 \oplus \mu,$$

where  $\mu$  is the orthogonal complement of  $B\mathcal{D}_2$  in  $\ker F_*$  and is invariant under  $J$ . Furthermore,

$$\begin{aligned} C\mathcal{D}_1 &= \mathcal{D}_1, B\mathcal{D}_1 = 0, C\mathcal{D}_2 \subset \mathcal{D}_2, \omega(\ker F_*) = \mathcal{D}_2 \\ \phi^2 + B\omega &= -id, C^2 + \omega B = -id, \omega\phi + C\omega = 0, BC + \phi B = 0. \end{aligned}$$

Define the tensors  $\mathcal{T}$  and  $\mathcal{A}$  by

$$\begin{aligned} \mathcal{A}_E F &= \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F \\ \mathcal{T}_E F &= \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F \end{aligned}$$

for vector fields  $E, F$  on  $M$ , where  $\nabla$  is the Levi-Civita connection of  $g_M$ . Define

$$\widehat{\nabla}_X Y := \mathcal{V}\nabla_X Y \quad \text{for } X, Y \in \Gamma(\ker F_*).$$

Then we easily have

**Lemma 3.3.** *Let  $(M, g_M, J)$  be a Kähler manifold and  $(N, g_N)$  a Riemannian manifold. Let  $F : (M, g_M, J) \mapsto (N, g_N)$  be a v-semi-slant submersion. Then we get*

(1)

$$\begin{aligned} \widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y &= \phi \widehat{\nabla}_X Y + B\mathcal{T}_X Y \\ \mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y &= \omega \widehat{\nabla}_X Y + C\mathcal{T}_X Y \end{aligned}$$

for  $X, Y \in \Gamma(\ker F_*)$ .

(2)

$$\begin{aligned} \mathcal{V}\nabla_Z BW + \mathcal{A}_Z CW &= \phi \mathcal{A}_Z W + B\mathcal{H}\nabla_Z W \\ \mathcal{A}_Z BW + \mathcal{H}\nabla_Z CW &= \omega \mathcal{A}_Z W + C\mathcal{H}\nabla_Z W \end{aligned}$$

for  $Z, W \in \Gamma((\ker F_*)^\perp)$ .

(3)

$$\begin{aligned} \widehat{\nabla}_X BZ + \mathcal{T}_X CZ &= \phi \mathcal{T}_X Z + B\mathcal{H}\nabla_X Z \\ \mathcal{T}_X BZ + \mathcal{H}\nabla_X CZ &= \omega \mathcal{T}_X Z + C\mathcal{H}\nabla_X Z \end{aligned}$$

for  $X \in \Gamma(\ker F_*)$  and  $Z \in \Gamma((\ker F_*)^\perp)$ .

**Theorem 3.4.** *Let  $F$  be a  $v$ -semi-slant submersion from an almost Hermitian manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$ . Then the slant distribution  $\mathcal{D}_2$  is integrable if and only if we obtain*

$$\mathcal{A}_X Y = 0 \quad \text{and} \quad PC(\nabla_X Y - \nabla_Y X) = 0$$

for  $X, Y \in \Gamma(\mathcal{D}_2)$

*Proof.* For  $X, Y \in \Gamma(\mathcal{D}_2)$  and  $Z \in \Gamma(\mathcal{D}_1)$ , assume that  $\mathcal{A}_X Y = \frac{1}{2}\mathcal{V}[X, Y] = 0$ , we obtain

$$\begin{aligned} g_M([X, Y], JZ) &= -g_M(J(\nabla_X Y - \nabla_Y X), Z) \\ &= -g_M(B\nabla_X Y + C\nabla_X Y - B\nabla_Y X - C\nabla_Y X, Z) \\ &= -g_M(C(\nabla_X Y - \nabla_Y X), Z). \end{aligned}$$

Therefore, we have the result.  $\square$

Similarly, we get

**Theorem 3.5.** *Let  $F$  be a  $v$ -semi-slant submersion from an almost Hermitian manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$ . Then the complex distribution  $\mathcal{D}_1$  is integrable if and only if we have*

$$\mathcal{A}_X Y = 0 \quad \text{and} \quad B(\nabla_X Y - \nabla_Y X) = 0$$

for  $X, Y \in \Gamma(\mathcal{D}_1)$ .

**Lemma 3.6.** *Let  $(M, g_M, J)$  be a Kähler manifold and  $(N, g_N)$  a Riemannian manifold. Let  $F : (M, g_M, J) \mapsto (N, g_N)$  be a  $v$ -semi-slant submersion. Then the complex distribution  $\mathcal{D}_1$  is integrable if and only if we get*

$$\mathcal{A}_X Y = 0 \quad \text{for } X, Y \in \Gamma(\mathcal{D}_1).$$

*Proof.* For  $X, Y \in \Gamma(\mathcal{D}_1)$  and  $Z \in \Gamma(\ker F_*)$ , assume that  $\mathcal{A}_X Y = 0$ , we have

$$\begin{aligned} g_M([X, Y], \omega Z) &= g_M([X, Y], JZ) = -g_M(J(\nabla_X Y - \nabla_Y X), Z) \\ &= -g_M(\mathcal{A}_X JY + \mathcal{H}\nabla_X JY - \mathcal{A}_Y JX - \mathcal{H}\nabla_Y JX, Z) \\ &= -g_M(\mathcal{A}_X JY - \mathcal{A}_Y JX, Z). \end{aligned}$$

Therefore, the result follows.  $\square$

In a similar way, we have

**Lemma 3.7.** *Let  $(M, g_M, J)$  be a Kähler manifold and  $(N, g_N)$  a Riemannian manifold. Let  $F : (M, g_M, J) \mapsto (N, g_N)$  be a  $v$ -semi-slant submersion. Then the slant distribution  $\mathcal{D}_2$  is integrable if and only if we obtain*

$$\mathcal{A}_X Y = 0 \quad \text{and} \quad P((\mathcal{A}_X B Y - \mathcal{A}_Y B X) + \mathcal{H}(\nabla_X C Y - \nabla_Y C X)) = 0$$

for  $X, Y \in \Gamma(\mathcal{D}_2)$ .

**Proposition 3.8.** *Let  $F$  be a  $v$ -semi-slant submersion from an almost Hermitian manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$ . Then we obtain*

$$C^2 X = -\cos^2 \theta X \quad \text{for } X \in \Gamma(\mathcal{D}_2),$$

where  $\theta$  denotes the  $v$ -semi-slant angle of  $\mathcal{D}_2$ .

*Proof.* Since

$$\cos \theta = \frac{g_M(JX, CX)}{|JX| \cdot |CX|} = \frac{-g_M(X, C^2X)}{|X| \cdot |CX|}$$

and  $\cos \theta = \frac{|CX|}{|JX|}$ , we have

$$\cos^2 \theta = -\frac{g_M(X, C^2X)}{|X|^2} \quad \text{for } X \in \Gamma(\mathcal{D}_2).$$

Hence,

$$C^2X = -\cos^2 \theta X \quad \text{for } X \in \Gamma(\mathcal{D}_2).$$

□

*Remark 3.9.* It is easy to see that the converse of Proposition 3.8 is also true.

Assume that the v-semi-slant angle  $\theta$  is not equal to  $\frac{\pi}{2}$  and define an endomorphism  $\hat{J}$  of  $(\ker F_*)^\perp$  by

$$\hat{J} := JP + \frac{1}{\cos \theta} CQ.$$

Then,

$$(1) \quad \hat{J}^2 = -id \quad \text{on } (\ker F_*)^\perp.$$

From (1), we have

**Theorem 3.10.** *Let  $F$  be a v-semi-slant submersion from an almost Hermitian manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$  with the v-semi-slant angle  $\theta \in [0, \frac{\pi}{2})$ . Then  $N$  is an even-dimensional manifold.*

**Proposition 3.11.** *Let  $F$  be a v-semi-slant submersion from a Kähler manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$ . Then the distribution  $\ker F_*$  defines a totally geodesic foliation if and only if*

$$\omega(\hat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + \mathcal{H} \nabla_X \omega Y) = 0 \quad \text{for } X, Y \in \Gamma(\ker F_*).$$

*Proof.* For  $X, Y \in \Gamma(\ker F_*)$ ,

$$\begin{aligned} \nabla_X Y &= -J \nabla_X JY = -J(\hat{\nabla}_X \phi Y + \mathcal{T}_X \phi Y + \mathcal{T}_X \omega Y + \mathcal{H} \nabla_X \omega Y) \\ &= -(\phi \hat{\nabla}_X \phi Y + \omega \hat{\nabla}_X \phi Y + B \mathcal{T}_X \phi Y + C \mathcal{T}_X \phi Y + \phi \mathcal{T}_X \omega Y + \omega \mathcal{T}_X \omega Y \\ &\quad + B \mathcal{H} \nabla_X \omega Y + C \mathcal{H} \nabla_X \omega Y). \end{aligned}$$

Thus,

$$\nabla_X Y \in \Gamma(\ker F_*) \Leftrightarrow \omega(\hat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + \mathcal{H} \nabla_X \omega Y) = 0.$$

□

Similarly, we have

**Proposition 3.12.** *Let  $F$  be a v-semi-slant submersion from a Kähler manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$ . Then the distribution  $(\ker F_*)^\perp$  defines a totally geodesic foliation if and only if*

$$\phi(\mathcal{V} \nabla_X BY + \mathcal{A}_X CY) + B(\mathcal{A}_X BY + \mathcal{H} \nabla_X CY) = 0 \quad \text{for } X, Y \in \Gamma((\ker F_*)^\perp).$$

**Proposition 3.13.** *Let  $F$  be a  $v$ -semi-slant submersion from a Kähler manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$ . Then the distribution  $\mathcal{D}_1$  defines a totally geodesic foliation if and only if*

$$\phi \mathcal{A}_X JY + B\mathcal{H}\nabla_X JY = 0 \text{ and } Q(\omega \mathcal{A}_X JY + C\mathcal{H}\nabla_X JY) = 0$$

for  $X, Y \in \Gamma(\mathcal{D}_1)$ .

*Proof.* For  $X, Y \in \Gamma(\mathcal{D}_1)$ , we get

$$\begin{aligned} \nabla_X Y &= -J\nabla_X JY = -J(\mathcal{A}_X JY + \mathcal{H}\nabla_X JY) \\ &= -(\phi \mathcal{A}_X JY + \omega \mathcal{A}_X JY + B\mathcal{H}\nabla_X JY + C\mathcal{H}\nabla_X JY). \end{aligned}$$

Hence,

$$\nabla_X Y \in \Gamma(\mathcal{D}_1) \Leftrightarrow \phi \mathcal{A}_X JY + B\mathcal{H}\nabla_X JY = 0 \text{ and } Q(\omega \mathcal{A}_X JY + C\mathcal{H}\nabla_X JY) = 0.$$

□

In a similar way, we obtain

**Proposition 3.14.** *Let  $F$  be a  $v$ -semi-slant submersion from a Kähler manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$ . Then the distribution  $\mathcal{D}_2$  defines a totally geodesic foliation if and only if*

$$\begin{aligned} \phi(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY) + B(\mathcal{A}_X BY + \mathcal{H}\nabla_X CY) &= 0 \\ P(\omega(\mathcal{V}\nabla_X BY + \mathcal{A}_X CY) + C(\mathcal{A}_X BY + \mathcal{H}\nabla_X CY)) &= 0 \end{aligned}$$

for  $X, Y \in \Gamma(\mathcal{D}_2)$ .

**Theorem 3.15.** *Let  $F$  be a  $v$ -semi-slant submersion from a Kähler manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$ . Then  $F$  is a totally geodesic map if and only if*

$$\begin{aligned} \omega(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y) &= 0 \\ \omega(\widehat{\nabla}_X BZ + \mathcal{T}_X CZ) + C(\mathcal{T}_X BZ + \mathcal{H}\nabla_X CZ) &= 0 \end{aligned}$$

for  $X, Y \in \Gamma(\ker F_*)$  and  $Z \in \Gamma((\ker F_*)^\perp)$ .

*Proof.* Since  $F$  is a Riemannian submersion, we obtain

$$(\nabla F_*)(Z_1, Z_2) = 0 \text{ for } Z_1, Z_2 \in \Gamma((\ker F_*)^\perp).$$

For  $X, Y \in \Gamma(\ker F_*)$ , we have

$$\begin{aligned} (\nabla F_*)(X, Y) &= -F_*(\nabla_X Y) = F_*(J\nabla_X(\phi Y + \omega Y)) \\ &= F_*(\phi \widehat{\nabla}_X \phi Y + \omega \widehat{\nabla}_X \phi Y + B\mathcal{T}_X \phi Y + C\mathcal{T}_X \phi Y + \phi \mathcal{T}_X \omega Y + \omega \mathcal{T}_X \omega Y \\ &\quad + B\mathcal{H}\nabla_X \omega Y + C\mathcal{H}\nabla_X \omega Y). \end{aligned}$$

Thus,

$$(\nabla F_*)(X, Y) = 0 \Leftrightarrow \omega(\widehat{\nabla}_X \phi Y + \mathcal{T}_X \omega Y) + C(\mathcal{T}_X \phi Y + \mathcal{H}\nabla_X \omega Y) = 0.$$

For  $X \in \Gamma(\ker F_*)$  and  $Z \in \Gamma((\ker F_*)^\perp)$ , we get

$$\begin{aligned} (\nabla F_*)(X, Z) &= -F_*(\nabla_X Z) = F_*(J\nabla_X(BZ + CZ)) \\ &= F_*(\phi \widehat{\nabla}_X BZ + \omega \widehat{\nabla}_X BZ + B\mathcal{T}_X BZ + C\mathcal{T}_X BZ + \phi \mathcal{T}_X CZ + \omega \mathcal{T}_X CZ \\ &\quad + B\mathcal{H}\nabla_X CZ + C\mathcal{H}\nabla_X CZ). \end{aligned}$$

Hence,

$$(\nabla F_*)(X, Z) = 0 \Leftrightarrow \omega(\widehat{\nabla}_X BZ + \mathcal{T}_X CZ) + C(\mathcal{T}_X BZ + \mathcal{H}\nabla_X CZ) = 0.$$

Since  $(\nabla F_*)(X, Z) = (\nabla F_*)(Z, X)$ , the result follows.  $\square$

Let  $F : (M, g_M) \mapsto (N, g_N)$  be a Riemannian submersion. The map  $F$  is called a Riemannian submersion *with totally umbilical fibers* if

$$(2) \quad \mathcal{T}_X Y = g_M(X, Y)H \quad \text{for } X, Y \in \Gamma(\ker F_*),$$

where  $H$  is the mean curvature vector field of the fiber.

Then we obtain

**Lemma 3.16.** *Let  $F$  be a v-semi-slant submersion with totally umbilical fibers from a Kähler manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$ . Then we have*

$$H \in \Gamma(\mathcal{D}_2).$$

*Proof.* For  $X, Y \in \Gamma(\mu)$  and  $W \in \Gamma(\mathcal{D}_1)$ , we get

$$\mathcal{T}_X JY + \widehat{\nabla}_X JY = \nabla_X JY = J\nabla_X Y = B\mathcal{T}_X Y + C\mathcal{T}_X Y + \phi\widehat{\nabla}_X Y + \omega\widehat{\nabla}_X Y.$$

Using (2), we easily obtain

$$g_M(X, JY)g_M(H, W) = -g_M(X, Y)g_M(H, JW).$$

Interchanging the role of  $X$  and  $Y$ , we get

$$g_M(Y, JX)g_M(H, W) = -g_M(Y, X)g_M(H, JW)$$

so that combining the above two equations, we have

$$g_M(X, Y)g_M(H, JW) = 0,$$

which means  $H \in \Gamma(\mathcal{D}_2)$ .  $\square$

**Corollary 3.17.** *Let  $F$  be a v-semi-slant submersion with totally umbilical fibers from a Kähler manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$  such that  $\mathcal{D}_1 = (\ker F_*)^\perp$ . Then each fiber is minimal.*

*Remark 3.18.* Let  $F$  be a v-semi-slant submersion from a Kähler manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$ . Then there is a distribution  $\mathcal{D}_1 \subset (\ker F_*)^\perp$  such that

$$(\ker F_*)^\perp = \mathcal{D}_1 \oplus \mathcal{D}_2, \quad J(\mathcal{D}_1) = \mathcal{D}_1,$$

and the angle  $\theta = \theta(X)$  between  $JX$  and the space  $(\mathcal{D}_2)_p$  is constant for nonzero  $X \in (\mathcal{D}_2)_p$  and  $p \in M$ , where  $\mathcal{D}_2$  is the orthogonal complement of  $\mathcal{D}_1$  in  $(\ker F_*)^\perp$ . Furthermore,

$$C\mathcal{D}_2 \subset \mathcal{D}_2, \quad B\mathcal{D}_2 \subset \ker F_*, \quad \ker F_* = B\mathcal{D}_2 \oplus \mu,$$

where  $\mu$  is the orthogonal complement of  $B\mathcal{D}_2$  in  $\ker F_*$  and is invariant under  $J$ . For the curvature tensor, it is sufficient to calculate the holomorphic sectional curvatures in a Kähler manifold.

Given a plane  $P$  being invariant by  $J$  in  $T_p M$ ,  $p \in M$ , there is an orthonormal basis  $\{X, JX\}$  of  $P$ . Denote by  $K(P)$ ,  $K_*(P)$ , and  $\widehat{K}(P)$  the sectional curvatures of the plane  $P$  in  $M$ ,  $N$ , and the fiber  $F^{-1}(F(p))$ , respectively, where  $K_*(P)$  denotes the sectional curvature of the plane  $P_* = \langle F_* X, F_* JX \rangle$  in  $N$ . Let  $K(X \wedge Y)$  be the sectional curvature of the plane spanned by the tangent vectors  $X, Y \in T_p M$ ,  $p \in M$ . Using both Corollary 1 of [14, p.465] and (1.28) of [7, p.13], we obtain the following :

(1) If  $P \subset (\mu)_p$ , then with some computations we have

$$K(P) = \widehat{K}(P) + |\mathcal{T}_X X|^2 - |\mathcal{T}_X JX|^2 - g_M(\mathcal{T}_X X, J[JX, X]).$$

(2) If  $P \subset (\mathcal{D}_2 \oplus B\mathcal{D}_2)_p$  with  $X \in (\mathcal{D}_2)_p$ , then we get

$$\begin{aligned} K(P) = & \sin^2 \theta \cdot K(X \wedge BX) + 2(g_M((\nabla_X \mathcal{A})(X, CX), BX) + g_M(\mathcal{A}_X CX, \mathcal{T}_{BX} X) \\ & - g_M(\mathcal{A}_{CX} X, \mathcal{T}_{BX} X) - g_M(\mathcal{A}_X X, \mathcal{T}_{BX} CX)) + \cos^2 \theta \cdot K(X \wedge CX). \end{aligned}$$

(3) If  $P \subset (\mathcal{D}_1)_p$ , then we obtain

$$K(P) = K_*(P) - 3|\mathcal{V}J\nabla_X X|^2.$$

#### 4. EXAMPLES

**Example 4.1.** Let  $(M, g_M, J)$  be an almost Hermitian manifold. Let  $\pi : TM \mapsto M$  be the natural projection. Then the map  $\pi$  is a v-semi-slant submersion such that  $\mathcal{D}_1 = (\ker \pi_*)^\perp$  [7].

**Example 4.2.** Let  $(M, g_M, J)$  be a  $2m$ -dimensional almost Hermitian manifold and  $(N, g_N)$  a  $(2m-1)$ -dimensional Riemannian manifold. Let  $F$  be a Riemannian submersion from an almost Hermitian manifold  $(M, g_M, J)$  onto a Riemannian manifold  $(N, g_N)$ . Then the map  $F$  is a v-semi-slant submersion such that

$$\mathcal{D}_1 = ((\ker F_*) \oplus J(\ker F_*))^\perp \quad \text{and} \quad \mathcal{D}_2 = J(\ker F_*)$$

with the v-semi-slant angle  $\theta = \frac{\pi}{2}$ .

**Example 4.3.** Define a map  $F : \mathbb{R}^6 \mapsto \mathbb{R}^4$  by

$$F(x_1, x_2, \dots, x_6) = (x_1, x_3 \sin \alpha - x_5 \cos \alpha, x_6, x_2),$$

where  $\alpha \in (0, \frac{\pi}{2})$ . Then the map  $F$  is a v-semi-slant submersion such that

$$\mathcal{D}_1 = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \rangle \quad \text{and} \quad \mathcal{D}_2 = \langle \frac{\partial}{\partial x_6}, \sin \alpha \frac{\partial}{\partial x_3} - \cos \alpha \frac{\partial}{\partial x_5} \rangle$$

with the v-semi-slant angle  $\theta = \alpha$ .

**Example 4.4.** Define a map  $F : \mathbb{R}^8 \mapsto \mathbb{R}^4$  by

$$F(x_1, x_2, \dots, x_8) = (x_4, x_3, \frac{x_5 - x_8}{\sqrt{2}}, x_6).$$

Then the map  $F$  is a v-semi-slant submersion such that

$$\mathcal{D}_1 = \langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \rangle \quad \text{and} \quad \mathcal{D}_2 = \langle \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_8} \rangle$$

with the v-semi-slant angle  $\theta = \frac{\pi}{4}$ .

**Example 4.5.** Define a map  $F : \mathbb{R}^{12} \mapsto \mathbb{R}^5$  by

$$F(x_1, x_2, \dots, x_{12}) = (x_2, \frac{x_5 + x_6}{\sqrt{2}}, \frac{x_7 + x_9}{\sqrt{2}}, \frac{x_8 + x_{10}}{\sqrt{2}}, x_1).$$

Then the map  $F$  is a v-semi-slant submersion such that

$$\mathcal{D}_1 = \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9}, \frac{\partial}{\partial x_8} + \frac{\partial}{\partial x_{10}} \rangle \quad \text{and} \quad \mathcal{D}_2 = \langle \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6} \rangle$$

with the v-semi-slant angle  $\theta = \frac{\pi}{2}$ .



**Example 4.6.** Define a map  $F : \mathbb{R}^{10} \mapsto \mathbb{R}^6$  by

$$F(x_1, x_2, \dots, x_{10}) = \left( \frac{x_3 - x_5}{\sqrt{2}}, x_6, \frac{x_7 + x_9}{\sqrt{2}}, x_8, x_1, x_2 \right).$$

Then the map  $F$  is a v-semi-slant submersion such that

$$\mathcal{D}_1 = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle \text{ and } \mathcal{D}_2 = \left\langle \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_9} \right\rangle$$

with the v-semi-slant angle  $\theta = \frac{\pi}{4}$ .

**Example 4.7.** Define a map  $F : \mathbb{R}^8 \mapsto \mathbb{R}^4$  by

$$F(x_1, x_2, \dots, x_8) = (x_1, x_3 \cos \alpha - x_5 \sin \alpha, x_2, x_4 \sin \beta + x_6 \cos \beta),$$

where  $\alpha$  and  $\beta$  are constant. Then the map  $F$  is a v-semi-slant submersion such that

$$\mathcal{D}_1 = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle \text{ and } \mathcal{D}_2 = \left\langle \cos \alpha \frac{\partial}{\partial x_3} - \sin \alpha \frac{\partial}{\partial x_5}, \sin \beta \frac{\partial}{\partial x_4} + \cos \beta \frac{\partial}{\partial x_6} \right\rangle$$

with the v-semi-slant angle  $\theta$  with  $\cos \theta = |\sin(\alpha - \beta)|$ .

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